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CHROMATIC NUMBERS OF COMPETITION GRAPHS

by

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Previous work on competition graphs has emphasized characterization, not only of the competition graphs themselves but also of those graphs whose competition graphs are chordal or interval. The latter sort of characterization is of interest when a competition graph that is easily colorable would be useful, e.g. in a scheduling or assignment problem. This leads naturally to the following question: Given a graph F, does the structure of G tell us anything about the chromatic number χ of the competition graph C(G)? We show that in some cases we can calculate this chromatic number exactly, while in others we can place tight bounds on the chromatic number.

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CHROMATIC NUMBERS OF COMPETITION GRAPHS

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Abstract: Previous work on competition graphs has emphasized characterization, not only of the competition graphs themselves but also of those graphs whose competition graphs are chordal or interval. The latter sort of characterization is of interest when a competition graph that is easily colorable would be useful, e.g. in a scheduling or assignment problem. This leads naturally to the following question: Given a graph G, does the structure of G tell us anything about the chromatic number χ of the competition graph $C(G)$? We show that in some cases we can calculate this chromatic number exactly, while in others we can place tight bounds on the chromatic number.

1. Preliminaries. The *competition graph* of a directed graph $D = (V, A)$ is the undirected graph $C(D) = (V, E)$ in which $xy \in E$ if and only if there exists $z \in V$ such that $(x, z), (y, z) \in A$. While the notion of competition graph was first reported in Cohen's [7] work on models of ecosystems, these graphs have since found a number of other applications, as reported by Lundgren [15], Kim [13], and Raychaudhuri and Roberts [27]. The distance $d_G(x, y)$ from x to y is the length of a shortest x, y-path in G; the subscript will be suppressed when there is no danger of confusion. The *two-step graph* of a graph $G = (V, E)$ is the graph $S_2(G) = (V, E')$, where $xy \in E'$ if and only if G contains a path of length two joining x and y; this has also been called the *neighborhood graph* by Boland, Brigham, and Dutton [2, 1, 4]. The *square* of G is the graph $G^2 = (V, \hat{E})$, where $xy \in \hat{E}$ if and only if $d(x, y) \leq 2$. Given an $m \times n$ $(0, 1)$-matrix M, the *row graph* of M is the graph G of order m in which the vertices correspond to the rows of M and in which vertices i and j are adjacent if and only if there exists some column k containing 1's in rows i and j. The column graph is similarly defined. We denote by $\Delta(G)$ and $\delta(G)$ the maximum and minimum degrees, respectively, among all vertices of G. By $\alpha(G)$ we denote the largest independent, i.e., pairwise nonadjacent, set of vertices of G, while by $\omega(G)$ we denote the order of the largest *clique*, or complete subgraph, of G. For graph-theoretic notation and terminology not defined in this paper, see Harary [11].

In some applications, such as scheduling or assignment problems, the digraphs whose competition graphs are under investigation happen to be symmetric. In these cases, it is simpler to view these digraphs as undirected graphs. The competition graph of an undirected graph $G = (V, E)$ is obtained by first replacing each edge $xy \in E$ with arcs (x, y) and (y, x) and then applying the previous definition. In earlier work with Maybee [17], two of the present authors showed that the competition graph $C(G)$ of a loopless graph is the two-step graph $S_2(G)$; Raychaudhuri and Roberts [27] showed that if G has a loop at each vertex, then $C(G) = H^2$, where H is obtained from G by removing

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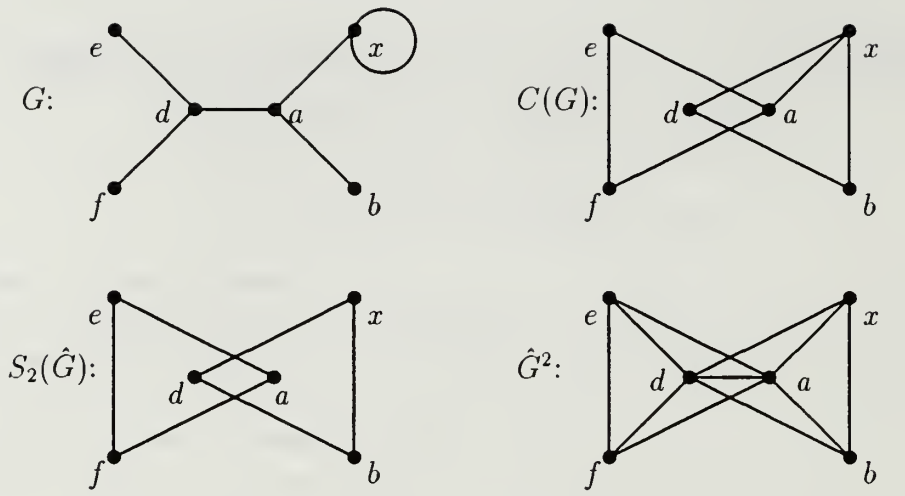

FIG. 1. *Graph G, with competition graph $C(G)$. Also shown are two-step graph $S_2(\hat{G})$ and square $\hat{G}^2$ of loopless graph $\hat{G}$ obtained from G.*

all loops. In previous work on competition graphs of symmetric digraphs, considerable emphasis has been on characterization of classes Γ of graphs whose competition graphs fall into some class Λ. For example, in the work of Raychaudhuri and Roberts [27, 26] these classes were $\Gamma = \{G|G$ is (unit) interval and has a loop at each vertex$\}$, and $\Lambda = \{G|G$ is (unit) interval$\}$. In our previous work [17] we set $\Gamma = \Lambda = \{G|G$ is loopless interval$\}$. Phelps [24] has set $\Gamma = \Lambda = \{G|G$ is chordal$\}$. The primary reason one seeks such characterizations is that applications of competition graphs to problems such as scheduling or channel assignment frequently involve vertex coloring. Since coloring is, in general, a hard problem, one seeks classes of graphs whose competition graphs have good coloring properties. This leads us to a related question that has not received much attention in the literature: given a graph G, what can be said of the chromatic number χ of the competition graph $C(G)$? Are there classes Γ of graphs with the property that if $G \in \Gamma$ then $\chi(C(G))$ can be bounded or exactly computed in terms of some parameter of G?

2. The Undirected Case. As a starting point, we have the obvious bound on the chromatic numbers of the two-step graph and square of a graph G given by $\Delta(G) \leq \chi(S_2(G)) \leq \chi(G^2)$. In an effort to improve on this, we begin by making an observation that generalizes aspects of earlier work, both ours [17, 18, 25, 22, 21] and that of Raychaudhuri and Roberts [27, 26]. Consider the graph G and its competition graph $C(G)$ of figure 1. Denote by $\hat{G}$ the graph obtained by deleting from G the loop xx. Included in figure 1 are $S_2(\hat{G})$ and $\hat{G}^2$. Note that $S_2(\hat{G}) \subseteq C(G) \subseteq \hat{G}^2$. This is always the case, as shown by the following lemma.

LEMMA 1. *Let G be a graph, and let $\hat{G}$ be the subgraph of G obtained by deleting loops. Then $S_2(\hat{G}) \subseteq C(G) \subseteq \hat{G}^2$.*

Proof: Let $xy \in E(S_2(\hat{G}))$. Then $x, y \in N(z)$ for some $z \in V$, so $xy \in C(G)$, whence $S_2(\hat{G}) \subseteq C(G)$. Now suppose that $xy \notin E(\hat{G}^2)$. Then $d_G(x,y) = d_{\hat{G}}(x,y) \geq 3$, so $xy \notin C(G)$, but then $C(G) \subseteq \hat{G}^2$. □

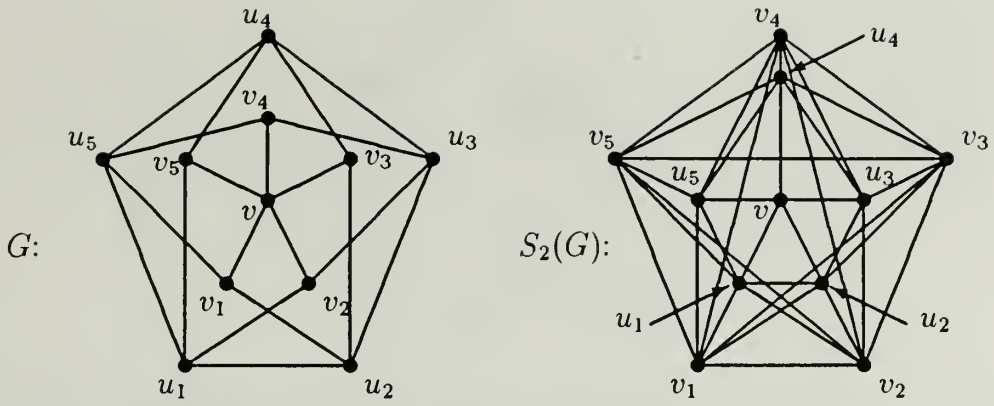

FIG. 2. *The Grötzsch graph G and its two-step graph. The square in this case is complete.*

This has a natural interpretation in terms of the chromatic number.

COROLLARY 1. *Let G be a graph, and let $\hat{G}$ be the subgraph of G obtained by deleting loops. Then $\chi(S_2(\hat{G})) \leq \chi(C(G)) \leq \chi(\hat{G}^2)$.*

Proof: Immediate. □

In the case depicted in figure 1, the bounds obtained differed by exactly one. It is easy to construct examples that show that this is not always the case. For example, consider the graph G of figure 2. This is the well-known Grötzsch graph (see, for instance, Bondy [3]), a 4-chromatic triangle-free graph. The two-step graph $S_2(G)$, also shown in figure 2, has chromatic number 6, while G^2 is isomorphic to K_{11} and so has chromatic number 11. Thus the bounds obtained by Corollary 1 are not particularly precise.

There is an obvious relationship between neighborhoods of (nonsimplicial) vertices in G and cliques in $C(G)$, from which we may deduce that $\Delta(G) \leq \omega(C(G))$. Moreover, we may replace $\chi(S_2(G))$ with $\omega(S_2(G))$ in the left-hand inequality of Corollary 1. However, we must further restrict our choice of G to make a similar substitution in the right hand inequality. Perfect graphs provide the necessary additional structure. There are several characterizations of perfect graphs. The characterization that we shall use here is the following: a graph G is said to be perfect if $\omega(G') = \chi(G')$ for all induced subgraphs G' of G. For other characterizations and much of the lore of perfect graphs, see Golumbic [10].

COROLLARY 2. *If $G = (V, E)$ and $\hat{G}$ are as described in Lemma 1, and if both $S_2(\hat{G})$ and $\hat{G}^2$ are perfect, then $\omega(S_2(\hat{G})) \leq \chi(C(G)) \leq \omega(\hat{G}^2)$.*

Proof: follows immediately from Corollary 1 and the definition of perfect graph. □

We are now able to obtain a result exemplified by the graphs in figure 1. The reader may verify that in that example we have $\chi(S_2(\hat{G})) = 3$ and $\chi(\hat{G}^2) = 4$.

THEOREM 1. *Let $G = (V, E)$ be a graph, and suppose that the subgraph $\hat{G}$ obtained by deleting loops is a tree. Then $\Delta(\hat{G}) \leq \chi(C(G)) \leq 1 + \Delta(\hat{G})$. Moreover, if G is loopless, then $\chi(C(G)) = \Delta(G)$.*

Proof: By [17], the maximal cliques of $S_2(\hat{G})$ correspond to the open neighborhoods of interior vertices of $\hat{G}$. So $\omega(S_2(\hat{G})) = \Delta(\hat{G})$. By a result of Raychaudhuri and

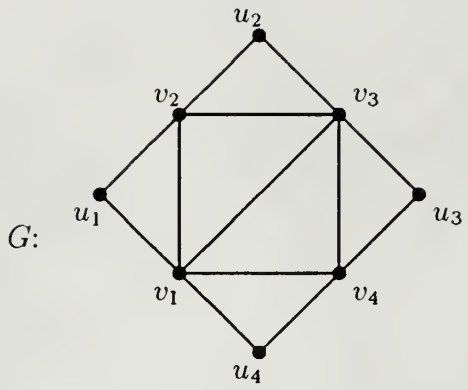

FIG. 3. *A 4-sun. Incomplete, since v_2v_4 is missing.*

Roberts [27], the maximal cliques of $\hat{G}^2$ correspond to the closed neighborhoods of interior vertices in $\hat{G}$. Thus $\omega(\hat{G}^2) = 1 + \Delta(\hat{G})$. Lundgren and Rasmussen [22] showed that if $\hat{G}$ is a tree then $S_2(\hat{G})$ is chordal; by a result of Chang and Nemhauser [6], $\hat{G}^2$ is chordal. Chordal graphs are perfect, so by Corollary 2 and the preceding observations we have $\Delta(\hat{G}) \leq \chi(C(G)) \leq 1 + \Delta(\hat{G})$. If G is loopless, then $C(G) = S_2(G)$, so we have $\chi(C(G)) = \Delta(G)$. □

Corollary 2, with its result concerning perfect graphs, is potentially useful only up to our ability to detect perfect graphs. Applying the definition alone is difficult, but certain classes of perfect graphs are easily recognized. One such class is that of chordal graphs. It is easy to construct chordal graphs whose two-step graphs are not chordal. For example, consider the graph G of figure 4. This graph is the simplest instance of a *bowtie*, where a bowtie is defined as a pair of triangles connected by a path of positive length. Similarly, it is a simple matter to construct a chordal graph G whose square is not chordal. For example, consider the "4-sun" G of figure 3. The vertices u_i, $i = 1, 2, 3, 4$ induce a 4-cycle in G^2. We can, however, provide the following sufficient condition for the square of a chordal graph to be chordal. The proof requires two additional definitions. The 4-sun of figure 3 is the case $n = 4$ of an object called an *n-sun*. An n-sun is defined as a chordal graph on $2n$ vertices $u_1, \ldots, u_n, v_1, \ldots, v_n$ in which the u_i are a stable set, the v_i are an n-cycle (not induced), and in which each vertex u_i is adjacent to vertices v_i and v_{i+1}, where the addition is cyclic. A *complete n-sun* is an n-sun in which $< v_1, v_2, \ldots, v_n >$ is complete. A *strongly chordal* graph G is a chordal graph with no induced n-sun, $n \geq 3$. These have been studied by Farber [8], Chang [5], Chang and Nemhauser [6], and Laskar and Shier [14], among others. Other names for n-suns are "trampoline", used by Farber, and "sunflower", used by Laskar and Shier. We will make use of the following lemma, which paraphrases part of a result of Chang and Nemhauser [6], found independently by Laskar and Shier [14], and which is stated without proof.

LEMMA 2. *The following statements are equivalent for any chordal graph G.*

1. *G^2 is chordal.*
2. *If G contains an induced n-sun, $n \geq 4$, defined on $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$, then $d_G(u_i, u_j) = 2$ for some $i < j$ such that $1 < |j - i| < n - 1$.*

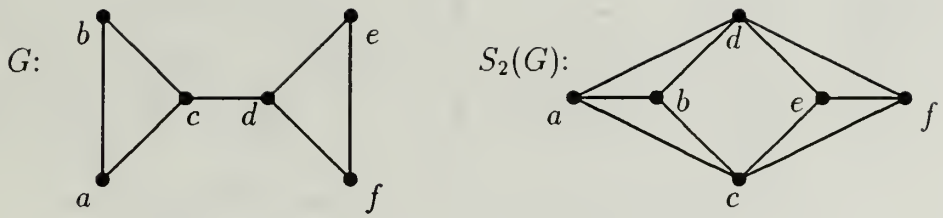

FIG. 4. *A bowtie G, with $S_2(G)$.*

Thus while G being strongly chordal is sufficient to guarantee that G^2 is chordal, there are ways to embed an n-sun in a graph G in such a way that G^2 is still chordal. Consider, for instance, the graph G of figure 3. Suppose that we introduce a new vertex, say x, and edges xu_1, xu_3. While $(G-x)^2$ is not chordal, since it contains $< u_1, u_2, u_3, u_4 >$ as an induced 4-cycle, it is easy to verify that G^2 is chordal. We may use Lemma 2 to prove another sufficient condition for the square of a chordal graph G to be chordal.

THEOREM 2. *Let $G = (V, E)$ be a chordal graph. If $S_2(G)$ is chordal, then G^2 is chordal.*

Proof: Suppose that both G and $S_2(G)$ are chordal. If G is strongly chordal, then by Lemma 2 G^2 is chordal, and we are done. Suppose, then, that G is not strongly chordal. Then G contains an induced n-sun H. If $n = 3$, then by Lemma 2 G^2 is chordal, so assume that $n \geq 4$ and that H is defined on $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ as described above. Since G contains H yet $S_2(G)$ is chordal, by Phelps [24] we know that $d_G(u_i, u_j) = 2$ for some $i < j$ with $1 < |j - i| < n - 1$, but then by Lemma 2 we know that G^2 is chordal and the proof is complete. □

Note that the statement of Lemma 2 leaves open the possibility that we might find a strongly chordal graph G whose square is chordal but whose two-step graph is not. Such graphs indeed exist. For example, consider the bowtie G of figure 4. The square of this graph is chordal, but we have already seen that its two-step graph is not. A precise statement of the conditions under which this occurs requires an additional definition, due to Phelps [24]. Given a graph $G = (V, E)$ and subgraph $H = (V', E')$ of G, and a set $R \subseteq \mathcal{P}(V')$, we say that H is *R-induced* if, whenever there exist $x \in V - V'$ and $y, z \in V'$ such that $xz, yz \in E$, then there must exist $S \in R$ such that $y, z \in S$. Given a bowtie B, let $T_1(B)$ and $T_2(B)$ denote the vertex sets of the two triangles in B. Given an n-sun G, let $\{u_1, \ldots, u_n\}$ denote the stable set and $W = \{v_1, \ldots, v_n\}$ denote the vertices of the n-cycle, as described earlier in the definition of n-sun. Phelps showed that a condition that is both necessary and sufficient for the two-step graph $S_2(G)$ of a chordal graph G to be chordal is that G contain no R-induced bowtie B with $R = \{T_1(B), T_2(B)\}$ and no R-induced n-sun, $n \geq 4$, with $R = \{\{W \bigcup u_i\} | i = 1, 2, \ldots, n\}$. In the sequel, we shall refer simply to R-induced bowties and R-induced n-suns; the reader should assume that the restrictions R are as stated above. Note that an R-induced n-sun is precisely an n-sun that violates condition (2) of Lemma 2.

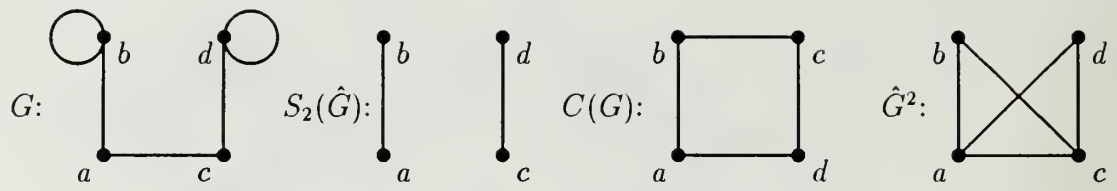

FIG. 5. *$\hat{G}$, $S_2(\hat{G})$, and $\hat{G}^2$ are chordal; $C(G)$ is not.*

Also left open by Lemma 2 is the possibility that $\hat{G}$, $S_2(\hat{G})$, and $\hat{G}^2$ are all chordal yet $C(G)$ is not. This, too, can occur. An instance of this is shown in figure 5.

Now suppose that we are able to determine, given some graph G, that Corollary 2 applies. In particular, suppose that we know that both $S_2(\hat{G})$ and $\hat{G}^2$ are chordal. The problem now is to bound $\chi(C(G))$ in terms of some parameter of $\hat{G}$. If we insist that G contain no six-cycle, a recent result by the present authors [16] shows that the maximal cliques in $S_2(\hat{G})$ correspond to the neighborhoods of the nonsimplicial vertices in $(\hat{G})$. Whether the closed or the open neighborhood of a nonsimplicial vertex x applies depends upon the sizes of cliques containing x. A similar result by the same authors [20] shows that the maximal cliques in $\hat{G}^2$ correspond to the maximal closed neighborhoods of these same vertices.

THEOREM 3. *Suppose that $\hat{G}$ is six-cycle-free and contains no R-induced bowtie or R-induced n-sun, $n \geq 4$. Then $\Delta(\hat{G}) \leq \chi(C(G)) \leq 1 + \Delta(\hat{G})$. Moreover, if every edge of $\hat{G}$ is contained in a triangle, then $\chi(C(G)) = 1 + \Delta(\hat{G})$.*

Proof: Since $\hat{G}$ contains no forbidden bowtie or n-sun, both $S_2(\hat{G})$ and $\hat{G}^2$ are chordal, so Corollary 2 applies and we have $\omega(S_2(\hat{G})) \leq \chi(C(G)) \leq \omega(\hat{G}^2)$. Since $\hat{G}$ is six-cycle-free, by the discussion above we have $\Delta(\hat{G}) \leq \omega(S_2(\hat{G})) \leq 1 + \Delta(\hat{G})$ and $\omega(\hat{G}^2) = 1 + \Delta(\hat{G})$, which establishes the first set of inequalities. If every edge of $\hat{G}$ is contained in a triangle, then $S_2(\hat{G}) = \hat{G}^2$, and the equality in that case follows immediately. □

Finally, we find a similar result that applies in the case where $\hat{G}$ is interval.

THEOREM 4. *Let G be a graph, with underlying loopless graph $\hat{G}$. If $\hat{G}$ is interval and contains no R-induced bowtie, then $\Delta(\hat{G}) \leq \chi(C(G)) \leq 1 + \Delta(\hat{G})$.*

Proof: Since $\hat{G}$ is interval, then by a result of Lundgren, Maybee and Rasmussen [18] the maximal cliques in $S_2(G)$ correspond to the maximal neighborhoods (either open or closed, depending on structure) of nonsimplicial vertices in $\hat{G}$, so $\Delta(\hat{G}) \leq \omega(S_2(\hat{G})) \leq 1 + \Delta(\hat{G})$. Since $\hat{G}$ is interval, then $\hat{G}$ is also strongly chordal, so has no induced n-suns. If in addition $\hat{G}$ contains no R-induced bowtie, then we know that $S_2((\hat{G}))$ is chordal. By a result of Raychaudhuri [26], the maximal cliques in $\hat{G}^2$ correspond to the maximal closed neighborhoods of nonsimplicial vertices in $\hat{G}$, so $\omega(\hat{G}^2) = 1 + \Delta(\hat{G})$. By a result of Raychaudhuri and Roberts [27], the square of an interval graph is interval. Since both chordal and interval graphs are perfect, the result follows from Corollary 2. □

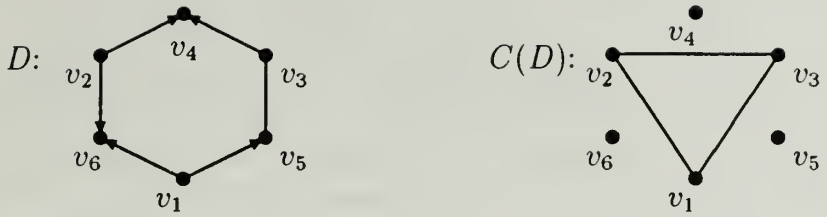

FIG. 6. *Digraph D has di-simplicial elimination ordering* $v_1, v_2, \ldots, v_6$.

3. The Directed Case. We now return briefly to the more general setting of directed graphs, where much of the existing work on competition graphs has been done. If D is a directed graph with adjacency matrix A, then the competition graph of D is simply the row graph of A. This gives us an immediate lower bound on $\chi(C(D))$, which is stated without proof in the following lemma.

LEMMA 3. *Let D be a directed graph on n vertices, without loops or multiple edges. Then* $\max_{v \in V}\{id(v)\} \leq \chi(C(D)) \leq n$.

To uncover anything less obvious, we must introduce a bit of relevant background material. If $D = (V, A)$ is a directed graph and $v \in V$, then we define Out(v) := {x ∈ V|(v,x) ∈ A}. Following Lundgren and Merz [19], say that a vertex $x \in V$ is *di-simplicial* in the digraph D if whenever there are distinct vertices $y, z, u, v \in V$ such that $(x,u),(y,u),(x,v),(z,v) \in A$, there exists a vertex $w \in V$ such that (y,w) and (z,w) are in A. The idea of a di-simplicial vertex in a digraph was first introduced by Hefner, et.al. [12]. Intuitively, we see that a vertex $x \in V$ is di-simplicial if the set of vertices with which vertex x competes are in mutual competition with one another. It is not hard to see that this set of vertices induces a complete subgraph in $C(D)$. See, for example, the digraph D of figure 6, in which each of v_1, v_2, and v_3 is nontrivially di-simplicial. This particular specimen also illustrates the following definition. We say that $v_1, v_2, \ldots, v_n$ is a *di-simplicial elimination ordering* if and only if v_i is di-simplicial in $D_i = (V, A_i)$ where A_i is the subgraph of A obtained by deleting all outarcs of $v_1, \ldots, v_{i-1}$. This is closely related to *perfect elimination orderings* for chordal graphs. See Golumbic [10] for more on perfect elimination orderings and their applications.

We then have the following characterization of digraphs whose competition graphs are chordal.

THEOREM 5. *(Lundgren & Merz [19]) Let D be a digraph. Then* $G = C(D)$ *is chordal if and only if D has a di-simplicial elimination ordering.*

Proof. First assume that G is chordal. Then G has a perfect elimination ordering $v_1, v_2, \ldots, v_n$. We claim that $v_1, v_2, \ldots, v_n$ is a di-simplicial elimination ordering for D. Consider a vertex v_i that is simplicial in $G_i = G - \{v_1, \ldots, v_{i-1}\}$. We claim v_i is di-simplicial in D_i. Suppose there exist $y, z, u, v \in V$ such that $(y,u),(v_i,u),(z,v),(v_i,v)$ all lie in A_i. Observe that $y, z \notin \{v_1, \ldots, v_{i-1}\}$, since these vertices have no outgoing arcs in D_i, so y and z are vertices in G_i. Since y and z compete with v_i at, respectively,

u and v, then (y, v_i) and (z, v_i) are edges in G_i. Since v_i is simplicial in G_i then y and z are adjacent in G_i. Thus y and z have common prey, i.e., there exists $w \in V$ such that (y, w) and (z, w) are arcs in A_i. Therefore v_i is di-simplicial in D_i.

Now assume that D has di-simplicial ordering $v_1, v_2, \ldots, v_n$. We claim $v_1, v_2, \ldots, v_n$ is a perfect elimination ordering for $C(D)$. Consider a vertex v_i that is di-simplicial in D_i. We must show that v_i is simplicial in G_i. Suppose there exist $y, z \in G_i$ such that $y, z \in N(v_i)$. Since $G_i \subseteq C(D)$, y and z have common prey with v_i. Thus there exist vertices u and v, where possibly $u = v$, such that $(y, u), (v_i, u), (z, v), v_i, v) \in A$. Since $y, z \in G_i$, then $y, z \notin \{v_1, \ldots, v_{i-1}\}$, so (y, u) and (z, v) are arcs in A_i. If $u = v$, then y and z are adjacent in $C(D)$, hence in G_i. If $u \neq v$ then, since v_i is di-simplicial in D_i, there exists $w \in V$ such that $(y, w), (z, w) \in A_i$, so y and z are adjacent in $C(D)$ and G_i. Thus v_i is simplicial in G_i, completing the proof. □

It is well known that a perfect elimination ordering can be found in linear time (Rose, Tarjan, and Leuker [28]). In the proof of the previous theorem we showed that a di-simplicial elimination ordering in the digraph is a perfect elimination ordering in the corresponding competition graph and vice versa. Thus a modification of the algorithms used to find a perfect elimination ordering in a chordal graph can be used to find a di-simplicial elimination ordering in the digraph. The algorithm which produces a perfect elimination ordering can also be slightly modified to produce a list of the maximal cliques and chromatic number of a chordal graph (Gavril [9]). Analogous results arise for digraphs with chordal competition graphs.

LEMMA 4. *Suppose that $D = (V, A)$ has a di-simplicial elimination ordering $v_1, v_2, \ldots, v_n$. For each vertex v_i, let $X_i = \{v_i\} \bigcup \{v_k | k > i$ and $\mathrm{Out(v_i)} \bigcap \mathrm{Out(v_k)} \neq \emptyset\}$. Let C be a maximal clique in $G = C(D)$, and let $i = \min\{k | v_k \in C\}$. Then $C = X_i$.*

Proof: First suppose that $z = v_k \in C$. If $k = i$, by definition, $v_k \in X_i$, so assume that $k \neq i$. By our choice of v_i, $k > i$. Since $z, v_i \in C$, then $\mathrm{Out(z)} \bigcap \mathrm{Out(v_i)} \neq \emptyset$, but then $z \in X_i$. Consequently $C \subseteq X_i$. Now since D has a di-simplicial elimination ordering, it follows that X_i induces a clique in G. If $X_i \neq C$, then X_i must properly contain C, but this is impossible since C is maximal. □

THEOREM 6. *Suppose that $D = (V, A)$ has a di-simplicial elimination ordering $v_1, v_2, \ldots, v_n$, and let X_i, $1 \leq i \leq n$, be defined as in the preceding lemma. Then*

$$\chi(C(D)) = \max_{1 \leq i \leq n} |X_i|$$

.

Proof: Since D has a di-simplicial elimination ordering, then by the preceding result we know that $G = C(D)$ is chordal. Since chordal graphs are perfect, it follows that $\chi(G) = \omega(G)$. By the preceding lemma, $\omega(G) = \max_{1 \leq i \leq n} |X_i|$. □

3.1. Circulant Tournaments. In pursuit of stronger results, we must begin with directed graphs that are highly structured. A *tournament* of order n is an oriented complete graph of order n. Following Moon [23], we define a *regular* tournament D of order n to be as regular as possible, i.e., if n is odd then D is regular in the usual sense,

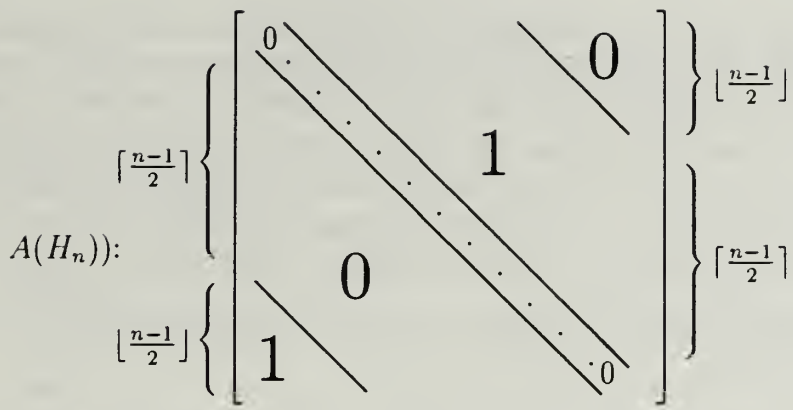

FIG. 7. *Adjacency matrix of H_n, the circulant tournament of order n.*

while if n is even then for every vertex v either $id(v) = \lceil \frac{n-1}{2} \rceil$ and $od(v) = \lfloor \frac{n-1}{2} \rfloor$, or vice versa. Among regular tournaments, one is of special interest in this paper. We define a *circulant tournament* to be a regular tournament whose adjacency matrix can be arranged in the form shown in figure 7. If the order n of the tournament is odd, then the matrix is a true circulant; in the even case, the first 1 to be cyclically shifted from the n^{th} column to the first is replaced with a 0, after which the shifts proceed in the usual way.

The following lemma will be useful in proving a subsequent result concerning the chromatic number of the circulant tournament of order n.

LEMMA 5. *Let H_n be the circulant tournament of order n, where $n > 4$. Then $\alpha(C(H_n)) = 2$.*

Proof: By relabeling the vertices of H_n as necessary, we may place $A(H_n)$ in the form shown in figure 7. A simple calculation shows that rows n and $\lceil \frac{n}{2} \rceil$ are orthogonal, so vertices n and $\lceil \frac{n}{2} \rceil$ have no common neighbors, whence these vertices are nonadjacent in $C(H_n)$. Thus $\alpha(C(H_n)) \geq 2$. One can examine the case $n = 6$ to verify that the lemma is true. For $n = 5$ and $n \geq 7$, since each row of $A(H_n)$ has at least $\lfloor \frac{n-1}{2} \rfloor$ ones and $3(\lfloor \frac{n-1}{2} \rfloor) > n$, if we choose any three rows i, j, and k, there must exist some column p such that at least two of rows i, j, and k have ones in column p. Thus $\alpha(C(H_n)) \leq 2$, and the result follows. □

Note that an immediate consequence of the preceding lemma is that, in any proper coloring of $C(H_n)$, no color class can contain more than two vertices. This is crucial in completing the proof of the following theorem.

THEOREM 7. *Let H_n be the circulant tournament of order n, where $n \geq 4$. Then*

$$\chi(C(H_n)) = \left\lceil \frac{n}{2} \right\rceil$$

Proof: The case $n = 4$ is easily verified, so assume that $n > 4$. We first show that $C(H_n)$ can be properly colored using $\lceil \frac{n}{2} \rceil$ colors. Relabel the vertices of H_n, if necessary, so that the adjacency matrix of H_n is in the form shown in figure 7. To

each vertex i, $i = 1, 2, \ldots, n$, assign color $i \pmod{\lceil \frac{n}{2} \rceil}$. Suppose two vertices v_i and v_j have the same color. Then $i \equiv j \pmod{\lceil \frac{n}{2} \rceil}$. Suppose v_i and v_j are adjacent in $C(H_n)$. Then there exists k such that v_i and v_j have arcs to v_k in the tournament and $a_{i,k} = a_{j,k} = 1$. Since $A(H_n)$ has the circular consecutive ones property for columns, there must be $\lceil \frac{n}{2} \rceil + 1$ ones in column k. But each column sum is either $\lfloor \frac{n-1}{2} \rfloor$ or $\lceil \frac{n-1}{2} \rceil$, a contradiction. Thus v_i and v_j are not adjacent in $C(H_n)$. Thus $\chi(C(H_n)) \leq \lceil \frac{n}{2} \rceil$. If fewer than $\lceil \frac{n}{2} \rceil$ colors are used, then at least one color must be used on at least three vertices, but this is impossible since, by the preceding lemma, each color class can have at most two members. Thus $\chi(C(H_n)) \geq \lceil \frac{n}{2} \rceil$, and the result follows. □

4. Avenues for Further Work. The results concerning tournaments leave a number of questions unanswered. For example, let $m(n)$ be the maximum number of cyclic triples in a tournament on n vertices. The minimum, of course, is 0, achieved by a transitive tournament. For all integers $0 \leq k \leq m$, there is a tournament on n vertices with exactly k cyclic triples. Do chromatic numbers of competition graphs of tournaments behave similarly? Is it the case that, for all integers $\lceil \frac{n}{2} \rceil \leq k \leq n - 1$, there exists a tournament D on n vertices with $\chi(C(D)) = k$? For that matter, if D is a tournament on n vertices, can we prove that $\lceil \frac{n}{2} \rceil \leq \chi(C(D))$?

In the undirected case, is the inequality, $\Delta(\hat{G}) \leq \chi(C(G)) \leq 1 + \Delta(\hat{G})$, satisfied by all graphs G whose underlying loopless graphs $\hat{G}$ are strongly chordal?

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